

# Measurement incompatibility and Channel Steering

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Incompatible measurements in quantum theory always lead to Einstein-Podolsky-Rosen (EPR)-Schrödinger steering. Channel steering which is a generalized notion of EPR-Schrödinger steering, has been introduced recently. Here we establish a connection between lack of joint measurability and channel steering.

PACS numbers: 03.65.Ud, 03.67.Ac

One of the important features of quantum theory is that not all measurements are compatible, i.e., they cannot be carried out simultaneously. Such a counter intuitive aspect makes quantum physics distinct from classical physics. This property is intimately connected to central tenets in the theory, such as Heisenberg's uncertainty principle [1], and Bohr's complementarity principle [2]. In the case of von Neuman measurements (projective measurements), compatibility is uniquely captured by the notion of commutativity. Non-commuting observables in quantum mechanics do not admit unambiguous joint measurement [3]. With the introduction of generalized measurements, i.e., positive operator-valued measures (POVMs) [4, 5], it was shown that observables which do not admit perfect joint measurement, may allow sufficiently fuzzy joint measurement [6]. Since for general measurements there is no unique notion of compatibility, here we focus on the well-defined criterion of joint measurability [7].

The optimal degree of unsharpness that guarantees joint measurement for all possible pairs of dichotomic observables of a theory may be considered as the degree of complementarity of the theory, which quantitatively binds the amount of optimal violation of the Bell-Clauser-Horne-Shimony-Holt (CHSH) inequality for any theory which satisfies the no-signaling principle [8]. It is also known that any set of two incompatible POVMs with binary outcomes may lead to a violation of the Bell-CHSH inequality [9]. However, this may not be extended to the general case of an arbitrary number of POVMs with arbitrarily many outcomes, since pairwise joint measurability does not imply full joint measurability in general [4]. On the other hand, it has been shown recently that measurement incompatibility in quantum theory always leads to EPR-Schrödinger steering [10]. It has been further shown by one of the authors of this article that the connection between measurement incompatibility and steering holds in a class of tensor product theories rather than just Hilbert space quantum mechanics [11].

Steering [12] refers to the scenario where one party, usually called Alice, wishes to convince the other party, called Bob, that she can steer the state at Bob's side by making measurements on her side. Steering has attracted much attention in recent years with the formulation of its information theoretic perspective [13], as well as the subsequent development [14] and applications [15] of steering inequalities. Experimental demonstrations of steering have followed using different settings and loophole free arrangements [16]. Practical applications of steering have been suggested in one-sided device-independent quantum key distribution [17] and sub-channel discrimination [18]. A resource theory of steering has also been proposed [19]. For the present purpose it is important to note that a set of POVMs in finite dimensions is not jointly measurable if and only if the set can be used to show the steerability of some quantum state [10].

Recently, the notion of steerability of quantum channels has been introduced by Piani [20], generalizing EPR-Schrödinger steerability. Consider that there is a quantum transformation (a quantum channel) from Charlie to Bob, which may be applied/used by Bob. Such transformation is in general noisy with information leaking to the environment (Alice). The relevant question here is the following: is Alice coherently connected to the input-output of the channel, or can she be effectively considered just a "classical bystander", with at most access to classical information about the transformation that affected the input of the channel? Steerability of a channel has been defined as the possibility for Alice to prove to Bob that she is not a "classical bystander", i.e., she is coherently connected with the input-output of the channel from Charlie to Bob. The way for Alice to prove so is by informing Bob of the choice of measurements performed by her and their outcomes. In this work we show that Alice is required to perform incompatible measurements in order to demonstrate channel steering.

We begin by first briefly discussing the mathematical framework of POVMs required for studying the notion of steerability for channels as introduced by Piani [20] as a generalization of the EPR-Schrödinger steering scenario. A POVM consists of a collection of operators  $\{M_{a|x}\}_a$  which are positive,  $M_{a|x} \geq 0 \forall a$ , and sum up to the identity,  $\sum_a M_{a|x} = \mathbf{1}$ . Here  $a$  denotes measurement outcome and  $x$  denotes measurement choice.

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A POVM may be realized physically by first letting the physical system interact with an auxiliary system and then measuring an ordinary observable on the auxiliary system. Let  $\{M_{\vec{a}}\}$  be a set of measurements with outcome  $\vec{a} = [a_{x=1}, a_{x=2}, \dots, a_{x=m}]$ , where  $a_x \in \{0, 1, \dots, n\}$  is the outcome of the  $x^{th}$  measurement. A set of  $m$  POVMs  $\{M_{a|x}\}_a$  is called jointly measurable if

$$M_{\vec{a}} \geq 0, \quad \sum_{\vec{a}} M_{\vec{a}} = \mathbf{1}, \quad \sum_{\vec{a} \setminus a_x} M_{\vec{a}} = M_{a|x} \quad \forall x, \quad (1)$$

where  $\vec{a} \setminus a_x$  stands for the elements of  $\vec{a}$  except for  $a_x$ . All POVM elements  $M_{a|x}$  are recovered as marginals of the observable  $M_{\vec{a}}$ .

The EPR-Schrödinger steering experiment can be completely characterized by specifying an ‘assemblage’  $\{\sigma_{a|x}\}_{a,x}$ , the set of sub-normalized states which Alice steers Bob into, given her choice of measurement  $x$  and outcome  $a$ . She can choose to perform one measurement from a set of  $m$  choices, each of which has  $n$  possible outcomes. The assemblage encodes the conditional probability distribution of her outcomes given her inputs  $p(a|x) = \text{Tr}(\sigma_{a|x})$ , as well as the conditional states prepared for Bob given Alice’s input and outcome  $\hat{\sigma}_{a|x} = \sigma_{a|x}/p(a|x)$ . All valid assemblages satisfy the consistency requirements,  $\sum_a \sigma_{a|x} = \sum_a \sigma_{a|x'}, \quad \forall x \neq x'$  and  $\text{Tr}(\sum_a \sigma_{a|x}) = 1$ . This encodes the fact that Alice cannot signal to Bob, and that without any knowledge about Alice, Bob still holds a valid quantum state. We denote this set of valid assemblages as  $\Sigma^S$ .

Assemblages which can be created via classical strategies (without using entanglement) are called unsteerable and denoted as  $\Sigma^{US}$ . Unsteerable assemblages can be expressed in the form

$$\sigma_{a|x} = \sum_{\lambda} p(a|x, \lambda) \sigma_{\lambda}, \quad \forall a, x \quad (2)$$

such that  $\text{Tr}(\sum_{\lambda} \sigma_{\lambda}) = 1$ ,  $\sigma_{\lambda} \geq 0 \quad \forall \lambda$ , where  $\lambda$  is a (classical) random variable held by Alice,  $p(a|x, \lambda)$  are conditional probability distributions for Alice, and  $\sigma_{\lambda}$  are the states held by Bob. Collection of unsteerable assemblages form a convex set [21]. Any assemblage that cannot be written in the above form is called steerable. For such assemblages there is no classical explanation as to how the different conditional states held by Bob could be prepared by Alice.

EPR and Schrödinger [12] observed that by performing measurements on her part of entangled quantum state shared with Bob, Alice can remotely prepare steerable assemblages on Bob’s side. Let us denote the measurement assemblage on Alice’s side as  $\{M_{a|x}^A\}_{a,x}$ , where  $M_{a|x}^A \geq 0 \quad \forall a, x$  and  $\sum_a M_{a|x}^A = \mathbf{1} \quad \forall x$ . This measurement assemblage whenever performed on Alice’s part of a bipartite quantum state  $\rho^{AB}$  shared between Alice and Bob, gives rise to the sub-states assemblage  $\{\sigma_{a|x}\}_{a,x}$  with  $\sigma_{a|x} = \text{Tr}_A(M_{a|x}^A \otimes \mathbf{1}^B \rho^{AB})$  and  $\sum_a \sigma_{a|x} = \text{Tr}_A(\rho^{AB})$  on Bob’s side. Though Schrödinger

pointed out steerability of bipartite pure entangled states in the very early days of quantum theory, it took a long time to establish that there exist mixed entangled states which exhibit this property [13].

A quantum channel  $\Lambda^{S \rightarrow S'} : \mathcal{D}(\mathcal{H}_S) \rightarrow \mathcal{D}(\mathcal{H}_{S'})$  is a completely-positive trace-preserving linear map [22], where  $S$  and  $S'$ , respectively, are the input and output quantum systems of the channel, and  $\mathcal{H}_*$  denotes the Hilbert space associated with the system.  $\mathcal{D}(\mathcal{H}_*)$  denotes the set of density operators acting on  $\mathcal{H}_*$ . We will denote a channel simply by  $\Lambda$ , whenever it is not required to specify the input-output system. The collection of completely-positive maps  $\Lambda_a$  is called an instrument  $\mathcal{I}$ , if  $\sum_a \Lambda_a$  is a channel. In such a case, each  $\Lambda_a$  is a subchannel, i.e., a completely positive trace-non-increasing linear map. A channel assemblage  $\mathcal{CA} := \{\mathcal{I}_x\}_x = \{\Lambda_{a|x}\}_{a,x}$  for a channel  $\Lambda$  is a collection of instruments  $\mathcal{I}_x$  for  $\Lambda$ , i.e.,  $\sum_a \Lambda_{a|x} = \Lambda$  for all  $x$ .

Consider a noisy quantum channel from  $C$  to  $B$ , ‘leaking’ information to the environment. Suppose that Alice has access to some part  $A$  of said environment. The situation can be modeled by quantum broadcast channels with one sender and two receivers [23]. This broadcast channel  $\Lambda^{C \rightarrow AB}$  is a channel extension of the given quantum channel  $\Lambda^{C \rightarrow B}$ . A channel extension  $\Lambda^{C \rightarrow AB}$  of a channel  $\Lambda^{C \rightarrow B}$  is called an incoherent extension if there exists an instrument  $\{\Lambda_{\lambda}^{C \rightarrow B}\}_{\lambda}$  with  $\sum_{\lambda} \Lambda_{\lambda}^{C \rightarrow B} = \Lambda^{C \rightarrow B}$ , and normalized (unit trace) quantum states  $\{\sigma_{\lambda}^A\}$ , such that

$$\Lambda^{C \rightarrow AB} = \sum_{\lambda} \Lambda_{\lambda}^{C \rightarrow B} \otimes \sigma_{\lambda}^A. \quad (3)$$

A channel extension is called a coherent extension if it is not incoherent.

We now address the issue as to under what circumstances is Alice coherently connected to the input-output of the channel. In such a case the map from  $C$  to  $AB$  is a coherent extension of the channel from  $C$  to  $B$ . Steerability of a channel extension is defined as the possibility for Alice to prove to Bob that she is not a classical bystander, or in other words that the leakage of information from  $C$  to  $A$  cannot be described in terms of a classical channel. As in the case of EPR-Schrödinger steering, here Alice is untrusted in the sense that we have no knowledge of either the state that Alice holds, or the measurements she performs. Note that one does not need to rely on the details/implementation of Alice’s measurements, i.e., the situation is device-independent on Alice’s side. Thus, the verification procedure does not require Bob to trust Alice’s measurement devices. Every choice of measurement by Alice corresponds to a different decomposition into subchannels of the channel used by Bob. A channel assemblage  $\mathcal{CA} = \{\Lambda_{a|x}\}_{a,x}$  is unsteerable if there exists an instrument  $\{\Lambda_{\lambda}\}_{\lambda}$ , and conditional probability distributions  $p(a|x, \lambda)$ , such that

$$\Lambda_{a|x} = \sum_{\lambda} p(a|x, \lambda) \Lambda_{\lambda}, \quad \forall a, x. \quad (4)$$

An unsteerable channel assemblage is denoted as  $\Lambda^{US}$ . A channel assemblage is steerable if it cannot be expressed in the above form. In the following we show that Alice is able to produce a steerable channel assemblage if and only if the measurements she performs are incompatible.

**Theorem:** The channel assemblage  $\{\Lambda_{a|x}^{C \rightarrow B}\}_{a,x}$  for a channel  $\Lambda = \Lambda^{C \rightarrow B}$ , with  $\Lambda_{a|x}^{C \rightarrow B} = \text{Tr}_A(M_{a|x}^A \Lambda^{C \rightarrow AB}[*])$ , is unsteerable for any channel extension  $\Lambda^{C \rightarrow AB}$  of  $\Lambda^{C \rightarrow B}$  if and only if the set of POVMs  $\{M_{a|x}^A\}_x$  applied by Alice on  $A$  is jointly measurable.

**Proof:** We first prove that joint measurability implies no channel steering. Let,  $\{M_{a|x}^A\}_{a,x}$  be jointly measurable, with the joint measurement operator denoted as  $M_{\vec{a}}^A$ , i.e.,

$$M_{\vec{a}}^A \geq 0, \quad \sum_{\vec{a}} M_{\vec{a}}^A = \mathbf{1}, \quad \sum_{\vec{a} \setminus a_x} M_{\vec{a}}^A = M_{a|x}^A,$$

where  $\vec{a} = [a_{x=1}, a_{x=2}, \dots, a_{x=m}]$ . Our aim is to show that the channel assemblage  $\{\Lambda_{a|x}^{C \rightarrow B}\}_{a,x}$  resulting from the measurement assemblage  $\{M_{a|x}^A\}_{a,x}$  on Alice's side for any channel extension (incoherent as well as coherent)  $\Lambda^{C \rightarrow AB}$  of  $\Lambda^{C \rightarrow B}$  is unsteerable, or in other words, there exists an instrument  $\{\Lambda_{\lambda}^{C \rightarrow B}\}_{\lambda}$ , with  $\sum_{\lambda} \Lambda_{\lambda}^{C \rightarrow B} = \Lambda^{C \rightarrow B}$ , and a conditional probability distribution  $p(a|x, \lambda)$ , such that

$$\Lambda_{a|x}^{C \rightarrow B} = \sum_{\lambda} p(a|x, \lambda) \Lambda_{\lambda}^{C \rightarrow B}, \quad \forall a, x. \quad (5)$$

Let,  $\lambda = \vec{a}$ ,  $\Lambda_{\lambda}^{C \rightarrow B} = \Lambda_{\vec{a}}^{C \rightarrow B} = \text{Tr}_A(M_{\vec{a}}^A \Lambda^{C \rightarrow AB}[*])$  and  $p(a|x, \lambda) = p(a|x, \vec{a}) = \delta_{a,a_x}$ . Clearly we have,

$$\begin{aligned} \sum_{\lambda} p(a|x, \lambda) \Lambda_{\lambda}^{C \rightarrow B} &= \sum_{\vec{a}} p(a|x, \vec{a}) \Lambda_{\vec{a}}^{C \rightarrow B} \\ &= \sum_{\vec{a}} \delta_{a,a_x} \text{Tr}_A(M_{\vec{a}}^A \Lambda^{C \rightarrow AB}[*]) \\ &= \text{Tr}_A\left(\sum_{\vec{a}} \delta_{a,a_x} M_{\vec{a}}^A \Lambda^{C \rightarrow AB}[*]\right) \\ &= \text{Tr}_A(M_{a|x}^A \Lambda^{C \rightarrow AB}[*]) \\ &= \Lambda_{a|x}^{C \rightarrow B} = (\Lambda^{C \rightarrow B})^{US}. \end{aligned} \quad (6)$$

In Ref.[20] it was shown that every unsteerable channel assemblage can be thought as arising from an incoherent channel extension. We can therefore conclude that by performing compatible measurements Alice cannot convince Bob that she is coherently connected with the input-output of the noisy channel applied by Bob.

We now prove the converse of the above result that if the channel assemblage  $\{\Lambda_{a|x}^{C \rightarrow B}\}_{a,x}$  for a channel  $\Lambda = \Lambda^{C \rightarrow B}$  with  $\Lambda_{a|x}^{C \rightarrow B} = \text{Tr}_A(M_{a|x}^A \Lambda^{C \rightarrow AB}[*])$  is unsteerable for any channel extension, then the measurement assemblage  $\{M_{a|x}^A\}_{a,x}$  applied by Alice is jointly measurable. In order to do so we use the Choi-Jamiołkowski representation [24] of channels. The Choi-Jamiołkowski isomorphic

operator of the channel  $\Lambda^{C \rightarrow AB}$  is given by

$$J_{C'AB}(\Lambda^{C \rightarrow AB}) := \Lambda^{C \rightarrow AB}[\psi_+^{CC'}], \quad (7)$$

where  $\psi_+^{CC'}$  is the density matrix corresponding to a fixed maximally entangled state of systems  $C$  and  $C'$ , with  $C'$  a copy of  $C$ .

The measurement assemblage  $\{M_{a|x}^A\}_{a,x}$  performed by Alice on her part of the extended channel  $\Lambda^{C \rightarrow AB}$  results in the channel assemblage  $\{\Lambda_{a|x}^{C \rightarrow B}\}_{a,x}$  for the channel  $\Lambda^{C \rightarrow B}$ , where  $\Lambda_{a|x}^{C \rightarrow B} = \text{Tr}_A(M_{a|x}^A \Lambda^{C \rightarrow AB}[*])$  with the Choi-Jamiołkowski operator  $J_{C'B}(\Lambda_{a|x}^{C \rightarrow B})$ . In [20], it has also been proved that the channel extension of a channel is steerable if and only if its Choi-Jamiołkowski operator is steerable. Now, if the channel assemblage  $\{\Lambda_{a|x}^{C \rightarrow B}\}_{a,x}$  is unsteerable, there exists an instrument  $\{\Lambda_{\lambda}^{C \rightarrow B}\}_{\lambda}$ , with  $\sum_{\lambda} \Lambda_{\lambda}^{C \rightarrow B} = \Lambda^{C \rightarrow B}$ , and conditional probability distribution  $p(a|x, \lambda)$ , such that  $\Lambda_{a|x}^{C \rightarrow B} = \sum_{\lambda} p(a|x, \lambda) \Lambda_{\lambda}^{C \rightarrow B}$  for all  $a, x$ . Clearly, the Choi-Jamiołkowski operator assemblage  $\{J_{C'B}(\Lambda_{a|x}^{C \rightarrow B})\}_{a,x}$  of the Choi-Jamiołkowski operator  $J_{C'B}(\Lambda^{C \rightarrow B})$  is also unsteerable, i.e.,

$$J_{C'B}(\Lambda_{a|x}^{C \rightarrow B}) = \sum_{\lambda} p(a|x, \lambda) J_{C'B}(\Lambda_{\lambda}^{C \rightarrow B}), \quad \forall a, x. \quad (8)$$

It now follows from the result of Refs.[10] that one can construct joint measurements for the measurement assemblage  $\{M_{a|x}^A\}_{a,x}$ . ■

To, summarize, in the present work we have studied the link between lack of joint measurability and channel steering. An important connection was established earlier between EPR-Schrödinger steering and the joint measurement of quantum observables. It was shown in Refs.[10] that incompatible measurements are needed to be performed for demonstrating EPR-Schrödinger steering. A generalization of the notion of EPR-Schrödinger steering has been introduced recently through the concept of channel steering [20]. Here one considers a noisy quantum transformation or channel between two parties (say, Charlie and Bob), leaking some information to the environment which is accessible to another party (say, Alice). The task of channel steering is for Alice to convince Bob that she is coherently connected to the input-output of the channel. In this work we have shown that Alice needs to perform incompatible measurements to succeed in her aim. By performing measurements that are jointly measurable Alice succeeds to produce only unsteerable channel assemblages of the noisy channel from Charlie to Bob. Our result establishes that non-joint measurability and channel steering imply each other. The connection between the two may have implications [25] for a resource theory of measurement incompatibility.

Acknowledgments: A.S.M. acknowledges support from the project SR/S2/LOP-08/2013 of DST, India.

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